Eleven Billion STS(19)s

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Joint work with

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[full screen: Ctrl-L]

Steiner triple systems

Steiner triple system (STS(v)): $(V, \mathcal{B})$, where V is a v-element set of points and $\mathcal{B}$ is a set of 3-element subsets of V (blocks (or lines or triples)) such that every unordered pair of points occurs in exactly one block.

Also called a balanced incomplete block design $(v, 3, 1)$.

Number of blocks: $b = |\mathcal{B}| = v(v-1)/6$.

There exists an STS(v) iff $v \equiv 1$ or $3 \pmod 6$.
Conjectured by Steiner in 1853,
Proved by Kirkman in 1847.

Small Steiner triple systems

Up to isomorphism there exist 1, 1, 1, 1, 2 and 80 Steiner triple systems of orders 1, 3, 7, 9, 13 and 15 respectively.

STS(1) : $V = \{0\}$, $\mathcal{B} = \varnothing$.

STS(3) : $V = \{0, 1, 2\}$, $\mathcal{B} = \{\{0, 1, 2\}\}$.

STS(7) : $V = \{0, 1, 2, 3, 4, 5, 6\}$,
$\mathcal{B} = \{\{0, 1, 2\}, \{0, 3, 4\}, \{1, 3, 5\}, \{2, 3, 6\}, \{1, 4, 6\}, \{2, 4, 5\},$
$\{0,5,6\}\}$. Projective plane of order 2.

STS(9) : $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$,
$\mathcal{B} =$ {{0,1,2}, {3,4,5}, {6,7,8}, {0,3,6}, {1,4,7}, {2,5,8},
{0,5,7}, {1,3,8}, {2,4,6}, {0,4,8}, {1,5,6}, {2,3,7}}.
Affine plane of order 3.

STS(19)s

Up to isomorphism there are 11084874829 STS(19)s.

Kaski & Östergård (2004);
Kaski, Östergård, Pottonen & Kiviluoto (to appear).

Available with (software to decompress them) from **Olli Pottonen**.

89 files; files 1 to 88: 125000000 systems each;
file 89: 84874829 systems.
Arranged in sets of 500 systems.
Total size 36.5 Gb (28 bits per system).

One year of CPU time = 2.8 ms per system.

Distinct STS(19)s: 1348410350618155344199680000.

Table 1: Automorphism group order

|Aut|	#	|Aut|	#	Serial number
1	11084710071	**19**	**1**	[79-27937664]
2	149522	24	11	
3	12728	32	3	
4	2121	54	2	
6	182	**57**	**2**	
8	101	96	1	[89-84864706]
9	19	108	1	[89-84864708]
12	37	144	1	[89-84864707]
16	13	**171**	**1**	[89-55913712]
18	11	432	1	[89-84864709]

The STS(19) with automorphism order 171 is set-wise 2-transitive and is generated by $\{0, 1, 4\}$, $x \mapsto 7x$, $x \mapsto x + 1$.
Usually called the **Netto** system.

The other three highlighted entries are cyclic,
[79-27937664], [57-117929611] and [82-36763749].

Table 4: Number of subsystems

# sub-STS(7)s	# subsystem STS(9)s 0	1	Total
0	10997902498	270784	10998173282
1	86101058	12956	86114014
2	572471	641	573112
3	11819	45	11864
4	2449	25	2474
6	75	5	80
12	2	1	3
Total	11084590372	284457	11084874829

The system with 12 sub-STS(7)s and one sub-STS(9) is the STS(19) with $|\mathrm{Aut}| = 432$.

Colourings of Steiner triple systems

Colour the points red, yellow, blue, etc.
No monochromatic blocks.

k-colourable means k colours are sufficient.
k-chromatic means k colours are sufficient, but not $k-1$.

The only 2-chromatic system is the STS(3).
There exist 3-chromatic STS(v)s for all $v \geq 7$
For $7 \leq v \leq 15$, all STS(v)s are 3-chromatic.
There exist 4-chromatic STS(21)s (Haddad, 1999).

It is known that all STS(19)s are 4-colourable
(ADF, Grannel & Griggs, 2003).

Big question : Are all STS(19)s 3-chromatic?

Result 1

All STS(19)*s are 3-chromatic.*

Proof (Finland)
Test each STS(19); CPU time: 1.5 days

STS(19) colouring patterns

Any 3-colouring must have one of the six patterns
$(7,6,6)$, $(7,7,5)$, $(8,6,5)$, $(8,7,4)$, $(9,5,5)$, $(9,6,4)$.

(c_R, c_Y, c_B)	x_{RYY}	x_{RBB}	x_{RRY}	x_{RRB}	x_{YYB}	x_{YBB}	x_{RYB}
(7, 6, 6)	$15-x$	x	$3+x$	$18-x$	x	$15-x$	6
(7, 7, 5)	$21-x$	$x-5$	$1+x$	$20-x$	x	$15-x$	5
(8, 6, 5)	$15-x$	$x-3$	$7+x$	$21-x$	x	$13-x$	4
(8, 7, 4)	$21-x$	$x-7$	$6+x$	$22-x$	x	$13-x$	2
(9, 5, 5)	$10-x$	$x-2$	$12+x$	$24-x$	x	$12-x$	1
(9, 6, 4)	$15-x$	$x-6$	$12+x$	$24-x$	x	$12-x$	0

Lemma 1

$(7,7,5) \to (7,6,6)$, $(8,6,5) \to (7,7,5)$ or $(7,6,6)$,
$(8,7,4) \to (7,7,5)$, $(9,5,5) \to (9,6,4)$ or $(8,6,5)$,
$(9,6,4) \to (8,6,5)$, $(8,7,4) \to (8,6,5)$,
$(9,5,5) \to (8,6,5)$, $(9,6,4) \to (9,5,5)$,
$(9,6,4) \to (8,7,4)$.

STS(19) colouring pattern combinations

$\mathcal{C}_1 = \{(7,6,6),(7,7,5)\}$,
$\mathcal{C}_2 = \{(7,6,6),(7,7,5),(8,6,5)\}$,
$\mathcal{C}_3 = \{(7,6,6),(7,7,5),(8,6,5),(8,7,4)\}$,
$\mathcal{C}_4 = \{(7,6,6),(7,7,5),(8,6,5),(9,5,5)\}$,
$\mathcal{C}_5 = \{(7,6,6),(7,7,5),(8,6,5),(8,7,4),(9,5,5)\}$,
$\mathcal{C}_6 = \{(7,6,6),(7,7,5),(8,6,5),(8,7,4),(9,5,5),(9,6,4)\}$,

$\mathcal{C}_7 = \{(7,6,6)\}$ (*balanced*),
$\mathcal{C}_8 = \{(7,6,6),(8,6,5)\}$,
$\mathcal{C}_9 = \{(7,6,6),(8,6,5),(9,5,5)\}$.

Combinations $\mathcal{C}_1$–$\mathcal{C}_6$ have been observed; but not $\mathcal{C}_7$–$\mathcal{C}_9$.

Question Does there exist a balanced STS(19)?

Theorem 4

Every STS(19) *has 3-colourings with patterns*
(i) (7,5,5) and
(ii) (8,6,5) with two exceptions.

Proof (me)
Test each STS(19); CPU time: 4 days.

The exceptions are the Netto system and another cyclic STS(19): [82-36763749].

'. . . Meanwhile, those of us who can compute can hardly be expected to keep writing papers saying 'I can do the following useless calculation in two seconds', and indeed what editor would publish them?' [Oliver Atkin]

Theorem 5

Every STS(19) *must have one of these combinations of 3-colouring patterns*

$\mathcal{C}_1 = \{(7,6,6),(7,7,5)\}$,
$\mathcal{C}_2 = \{(7,6,6),(7,7,5),(8,6,5)\}$,
$\mathcal{C}_3 = \{(7,6,6),(7,7,5),(8,6,5),(8,7,4)\}$,
$\mathcal{C}_4 = \{(7,6,6),(7,7,5),(8,6,5),(9,5,5)\}$,
$\mathcal{C}_5 = \{(7,6,6),(7,7,5),(8,6,5),(8,7,4),(9,5,5)\}$,
$\mathcal{C}_6 = \{(7,6,6),(7,7,5),(8,6,5),(8,7,4),(9,5,5),(9,6,4)\}$.

Proof

Corollary of Lemma 1 and Theorem 4.

Independent sets

An *independent set* in an STS(v) is a block-free subset of the point set.

A 3-colouring is a partitioning into 3 independent sets.

A *maximal independent set* is an independent set that cannot be increased to a bigger independent set by adding a point.

A *maximum independent set* is an independent set of the largest size.

There exist an STS(19) with maximum independent set of size m iff $m = 7, 8, 9$ or 10.

10 occurs precisely when the system has a sub-STS(9).

Table 7 CPU time: 12 weeks.

	Maximum independent set size				
Pattern	7	8	9	10	Total
$\mathcal{C}_1$	2	0	0	0	2
$\mathcal{C}_2$	0	53680512	2650830	1241	56332583
$\mathcal{C}_3$	0	10079422375	421936849	283216	10501642440
$\mathcal{C}_4$	0	0	2912144	0	2912144
$\mathcal{C}_5$	0	0	464995662	0	464995662
$\mathcal{C}_6$	0	0	58991998	0	58991998
	2	10133102887	951487483	284457	11084874829

$\mathcal{C}_1 = \{(7,6,6),(7,7,5)\}$,
$\mathcal{C}_2 = \{(7,6,6),(7,7,5),(8,6,5)\}$,
$\mathcal{C}_3 = \{(7,6,6),(7,7,5),(8,6,5),(8,7,4)\}$,
$\mathcal{C}_4 = \{(7,6,6),(7,7,5),(8,6,5),(9,5,5)\}$,
$\mathcal{C}_5 = \{(7,6,6),(7,7,5),(8,6,5),(8,7,4),(9,5,5)\}$,
$\mathcal{C}_6 = \{(7,6,6),(7,7,5),(8,6,5),(8,7,4),(9,5,5),(9,6,4)\}$.

Column 10 total 284457 agrees with the sub-STS(9) number of Stinson & Seah (1985).

Zeros in rows $\mathcal{C}_4$–$\mathcal{C}_6$ can be deduced without extensive computation.

Chromatic index

Colour the *blocks*.
Two intersecting blocks must not get the same colour.

A *resolvable* STS(v) has chromatic index $(v-1)/2$,
smallest possible.

STS(19)s are not resolvable; so the smallest possible chromatic index is 10.

Systems are known to exist with chromatic indices 10, 11 and 12.

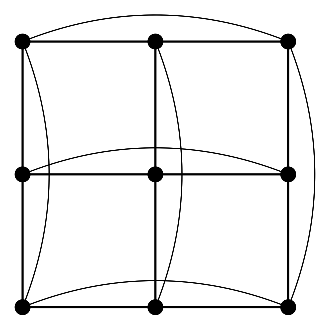
Theorem 6

11084870752 STS(19)*s have chromatic index* 10*;*
4075 STS(19)*s have chromatic index* 11*;*
two STS(19)*s have chromatic index* 12*.*

Proof
Test each STS(19); CPU time: 8 years.

Existential closure

2-existentially closed (2-e-c) graph: for any two vertices x, y, we can find vertices a, b, c, d such that $a \not\sim x$ and $a \not\sim y$, $b \sim x$ and $b \not\sim y$, $c \sim x$ and $c \not\sim y$, $d \sim x$ and $d \sim y$.



n-existentially closed (n-e-c) graph: for every set S of vertices with $|S| = n$, for every subset T of S, there exists a vertex t which is joined to every vertex of T but none of $S \setminus T$.

Paley graphs with $q \geq 29$ are 3-e-c. Vertex set is GF(q), $q \equiv 1 \pmod 4$, $x \sim y$ iff $|x - y| = \square$.

Block intersection graph

Block-intersection graph of an STS(v):
vertices = blocks; $a \sim b$ iff blocks a and b intersect.

Strongly regular $\left(\frac{v(v-1)}{6}, \frac{3(v-3)}{2}, \frac{v+3}{2}, 9\right)$.

The block intersection graph of the STS(3) is K_1.
The block intersection graph of the STS(7) is K_7.
The block intersection graph of the STS(9) is $K_{3,3,3,3}$.

Existentially closed block intersection graphs

For $v \geq 13$, any STS(v) has a 2-e-c block intersection graph.

For $v \geq 19$, the block intersection graph of an STS(v) is 3-e-c closed iff
there is no sub-STS(7),
there is no sub-STS(9) and
every set of three parallel blocks ($\equiv$) has a cross-link.

Are there any STS(v)s with 3-e-c block intersection graph?

$v \leq 15$: **no**; simple computation.
$v \geq 37$: **no**; count $\equiv$ and $\equiv$ with cross-link.
$v \geq 31$: **no**; argument involving independent sets.
$v = 27$: **no**; as $v \geq 31$ plus a little more work.
$v = 25$: **no**; as $v = 27$ plus a lot more work.
$v = 21$: **mystery**—no idea.
$v = 19$: **YES!!**; two (ADF, Grannell & Griggs, 2005).
Are there any more?

Theorem 8

The number of STS(19)*s with 3-existentially closed block-intersection graph is* 2*.*

Proof
Test each STS(19).

One of the two is cyclic [57-117929611];
its graph is on the cover of M500 **210**.
The other has $|\mathrm{Aut}| = 8$ [70-107745605].

The cycle graph, uniformity and perfection

Let x, y be points of an STS(v) with point set V.
Let z be the third point in the block containing $\{x, y\}$.
The *cycle graph* $G_{x,y}$ has vertices $V \setminus \{x, y, z\}$, and
$a \sim b$ iff either $\{a, b, x\}$ or $\{a, b, y\}$ is a block.

$G_{x,y}$ consists of cycles of even lengths ≥ 4.
The lengths partition $v - 3$.

A *uniform* STS(v) has all cycle graphs the same.

A *perfect* STS(v) has all cycle graphs C_{v-3}.

14 perfect STS(v) known; $v =$ 7, 9, 25, 33, 79, 139, 367, 811, 1531, 25771, 50923, 61339, 69991, 135859.

There is no perfect STS(19) (Kaski, 2005).

One uniform STS(19) is known to exist.
Are there any more?

Theorem 2

There is exactly one uniform STS(19).

Proof
Test each STS(19); CPU time: 2 weeks.

It's the Netto system.

Cycle graph patterns; CPU time: 3 weeks.

7 cycle graph types, 128 combinations.

1	{4,4,4,4}	2	{4,4,8}	3	{4,6,6}	4	{4,12}
5	{6,10}	6	{8,8}	7	{16}		

5	1	57	5	357	1
347	1	134	3	457	17
567	2585	1347	5	3567	125
2457	255	3457	259	4567	5009893
12467	1	13467	2	23567	10
12347	39	12457	56	13457	89
23457	46863	14567	135588	24567	75786636
34567	174351058	123467	15	123457	51146
124567	8658874	134567	11039468	234567	8685731027
1234567	2124060807				

k	4	6	8	10	12	16
k-cycle-free	2591	1	381	66	2727	4

Sparseness

A Steiner triple system S is called *k-sparse* if for $4 \leq n \leq k$, every configuration in S of n blocks spans at least $n+3$ points.

The terminology originates from a conjecture of Erdős: for every integer $k \geq 4$, there exists a k-sparse STS(v) for all sufficiently large admissible v.

An n-block configurations is dense if it has ≥ 4 blocks and $\leq n+2$ points. k-sparseness occurs if there are no dense configurations of $\leq k$ blocks.

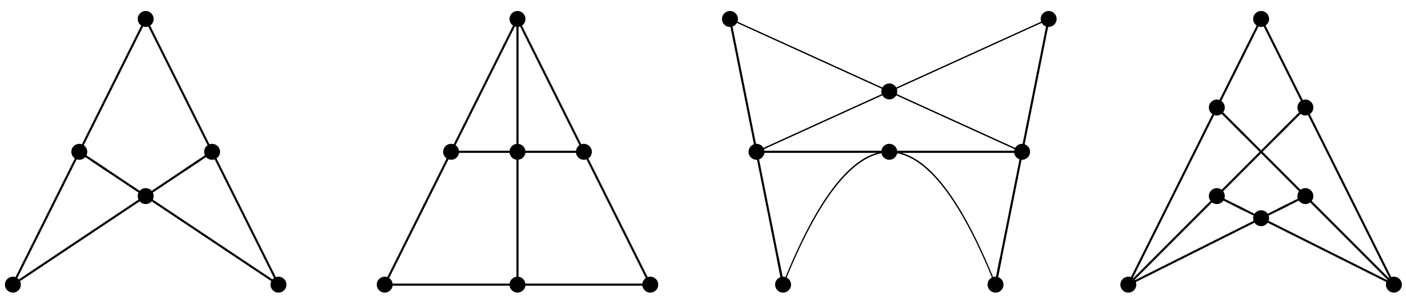
For practical purposes, k-sparseness is equivalent to the avoidance of certain n-block, $(n+2)$-point configurations for $4 \leq n \leq k$.

Dense configurations and sparseness

Pasch configuration (4-cycle): 4 blocks 6 points.
4-sparseness ≡ Pasch-free.

Mitre configuration: 5 blocks 7 points.
5-sparseness ≡ Pasch- and mitre-free.

Two 6-block, 8-point configurations: 6-cycle & crown.
6-sparseness ≡ Pasch-, mitre-, 6-cycle- and crown-free.



Pasch mitre crown 6-cycle

Theorem 1

The numbers of 4-sparse, 5-sparse and 6-sparse STS(19)*s are* 2591, 1 *and* 0 *respectively.*

Proof (Kaski & Östergård, 2004)
The 2591 4-sparse STS(19)s were made available by Kaski and Östergård as a result of their original enumeration. It is a simple exercise to test them for 5- and 6-sparseness. CPU time: 0 days.

The 5-sparse STS(19) is the Netto system.

Configurations

Configuration : subset of the block set of an STS(v).

Generator : configuration with no points of degree 1.

Let $\mathcal{C}$ be a k-block configuration, let S be an STS(v) and let $n(\mathcal{C}, S)$ denote the number of $\mathcal{C}$s that occur in S. Then

$$n(\mathcal{C}, S) \;=\; P(v) + \sum_{\mathcal{G}} Q_{\mathcal{G}}(v)\, n(\mathcal{G}, S),$$

where $\mathcal{G}$ runs through all generators of up to k blocks, and P and $Q_{\mathcal{G}}$ are polynomials in v.

The sum vanishes when $k \leq 3$.

Generators

4 blocks: one generator out of 16 configurations : Pasch.

5 blocks: one out of 56 : mitre.

6 blocks: 5 out of 282 : crown, 6-cycle,
semihead, window frame, prism.

7 blocks: 19 / 1865.

8 blocks: 153 / 17100.

9 blocks: 1615 / 207697.

10 blocks: 25180 / 3180571.

11 blocks: 479238 / ?.

12 blocks: 10695820 / ?.

Result 5 : Pasches (Kaski & Östergård, 2004)
2591 STS(19)s have none, 35758 STS(19)s have 1, etc.

Result 9 : mitres
4 STS(19)s have none, 11 STS(19)s have 3, etc.

Result 10 : 6-cycles
1 STS(19) has none [66-123826665], 1 has 2, etc.

Result 11 : crowns
4 STS(19)s have none, 8 STS(19)s have 24, etc.

Result 7 : window frames ⊞
1 STS(19) has 21, 1 STS(19) has 22, etc.
Window frames seem to be unavoidable generally.

Result 8 : prisms
1 STS(19) has 171, 1 STS(19) has 189, etc.

Conclusion

The STS(19)s that have non-trivial automorphism group are few in number and have been available for study for a number of years. Indeed, this would be a good place to try to find a system with a particular property.

But the recent efforts of Kaski, Östergård, Pottonen & Kiviluoto have now provided the STS(19)s that have *trivial* automorphism group. I am hoping that somewhere amongst these 11084710071 systems there exist waiting to be discovered one or two STS(19)s with truly remarkable properties. However, I'm not sure that this is likely to happen; so for the present I shall offer the following.

Conjecture

Almost all interesting Steiner triple systems of order 19 come from amongst the 164758 systems that have non-trivial automorphism group.