Geometric Graph Designs

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Joint work with

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G designs

Let G be a simple graph.
A G *design of order* n is a partitioning of the edges of a complete graph K_n into copies of graphs isomorphic to G.

(i) $n = 1$ or $n \geq |V(G)|$,
(ii) $n(n-1) \equiv 0 \pmod{2|E(G)|}$,
(iii) $n - 1 \equiv 0 \pmod{d}$,

where d is the GCD of the vertex degrees of G.

The problem: Determine the *spectrum* of G: $\{n : \exists$ a G design of order $n\}$.

Given G, a G design exists for all sufficiently large n satisfying (ii) and (iii) (R. M. Wilson, 1976).
The determination of the spectrum of G is therefore a finite problem.
Wilson's direct construction: There exists a G design of order $q > |E(G)|^{|V(G)|^2}$ whenever q is a prime power satisfying (ii) and (iii).

Examples

A K_k design of order n is a Steiner system $S(2, k, n)$.

The spectrum is known for $k = 2$ (trivial), $k = 3$ (Kirkman, 1847), and $k = 4$ & 5 (Hanani, 1961). Conditions (i)–(iii) are sufficient.

For larger k the results are incomplete.

A K_k design of order k^2, also called an *affine plane* of order k, exists whenever k is a prime power.

Survey article

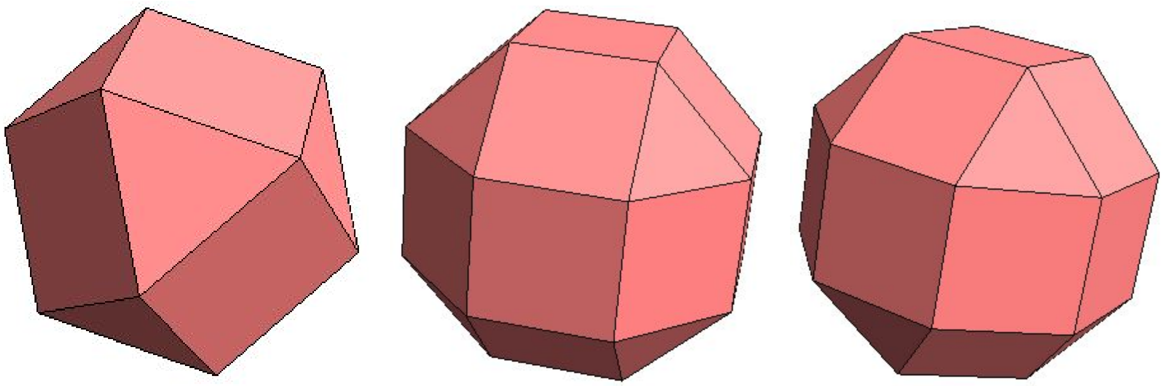
Adams, Bryant & Buchanan,
A survey on the existence of G designs, *J. Combin. Des.* (2008).

G-designs web site managed by Darryn Bryant and Tom McCourt

`http://wiki.smp.uq.edu.au/G-designs`

Trees, cycles, matchings, paths, stars, cubes, complete bipartite graphs, even graphs, theta graphs, . . . , and graphs of geometric solids.

Archimedean graphs

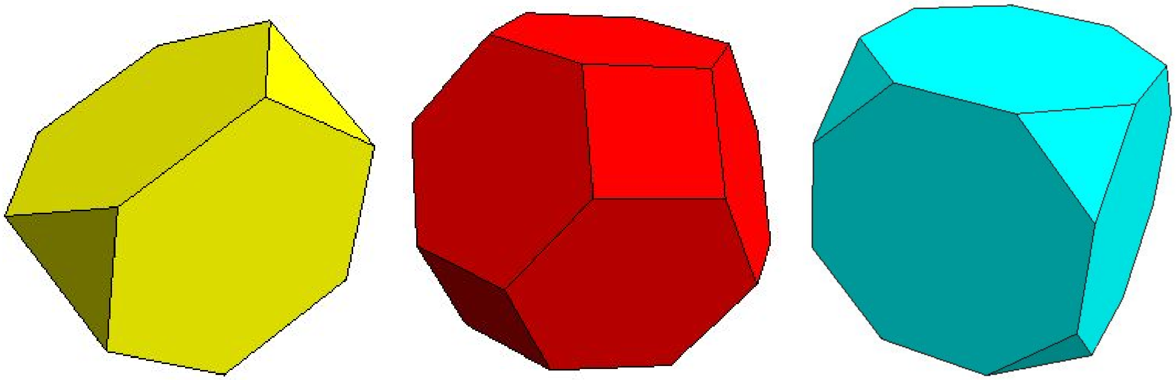


Cuboctahedron $d = 4$, $v = 12$, $e = 24$, $\chi = 3$,
$n \equiv 1, 33 \pmod{48}$
Done: Grannell, Griggs & Holroyd (2000).

Rhombicuboctahedron $d = 4$, $v = 24$, $e = 48$, $\chi = 3$,
$n \equiv 1, 33 \pmod{96}$
(Small rhombicuboctahedron)
Done: TF, Griggs & Holroyd (2010).

Pseudo-rhombicuboctahedron $d = 4$, $v = 24$,
$e = 48$, $\chi = 3$, $n \equiv 1, 33 \pmod{96}$
(Elongated square gyrobicupola, J_{37}, the $13\frac{1}{2}$th Archimedean solid)
Done: TF & Griggs (2011+).

Archimedean graphs



Truncated tetrahedron

$d = 3$, $v = 12$, $e = 18$, $\chi = 3$, $n \equiv 1$, 28 (mod 36)

Truncated octahedron

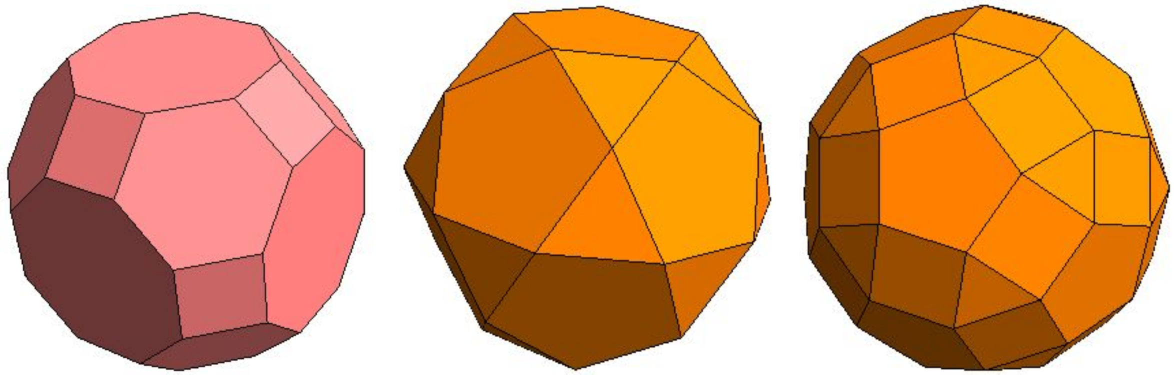
$d = 3$, $v = 24$, $e = 36$, $\chi = 2$, $n \equiv 1$, 64 (mod 72)

Truncated cube

$d = 3$, $v = 24$, $e = 36$, $\chi = 3$, $n \equiv 1$, 64 (mod 72)

All done: TF, Griggs & Holroyd (2011).

Archimedean graphs: residue class 1 (mod $2e$)



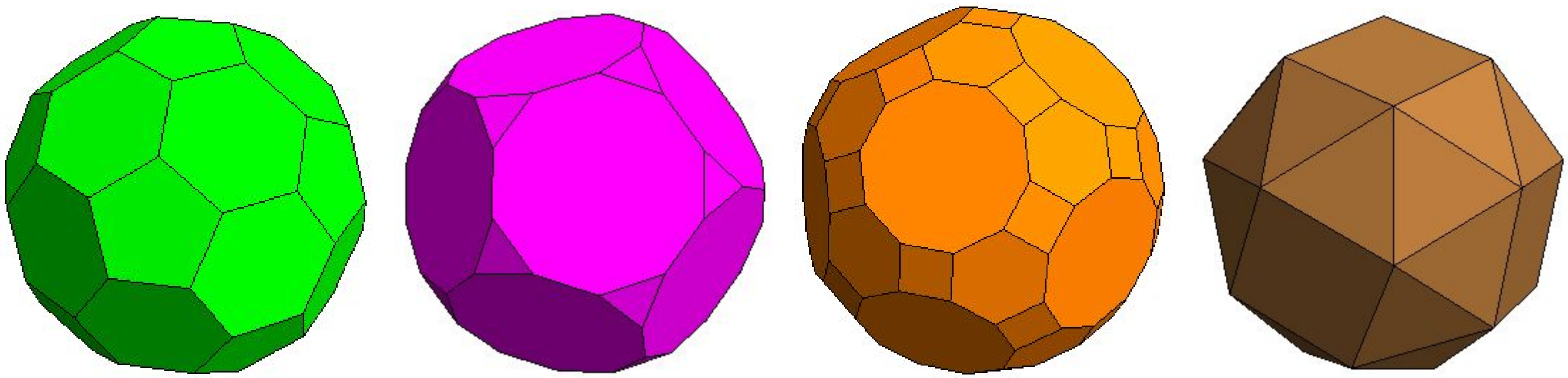
Truncated cuboctahedron $d = 3$, $v = 48$, $e = 72$, $\chi = 2$, $n \equiv 1$, 64 (mod 144)
(Great rhombicuboctahedron)

Icosidodecahedron $d = 4$, $v = 30$, $e = 60$, $\chi = 3$, $n \equiv 1$, 25, 81, 105 (mod 120), $n \neq 25$

Rhombicosidodecahedron $d = 4$, $v = 60$, $e = 120$, $\chi = 3$, $n \equiv 1$, 81, 145, 225 (mod 240)
(Small rhombicosidodecahedron)

Residue class 1 (mod $2e$) done; TF & Griggs (2011+).

Archimedean graphs: residue class 1 (mod $2e$)



Truncated icosahedron $d = 3$, $v = 60$,
$e = 90$, $\chi = 3$, $n \equiv 1$, 100, 136, 145 (mod 180)
(Football)

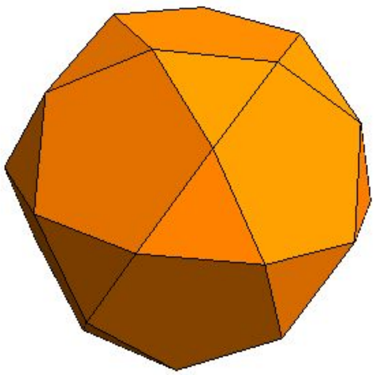
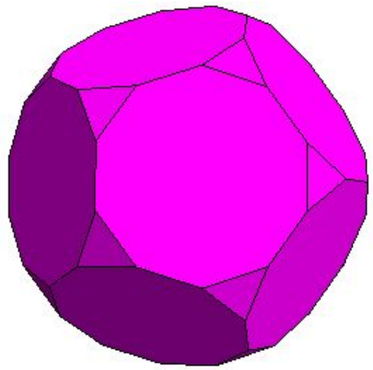
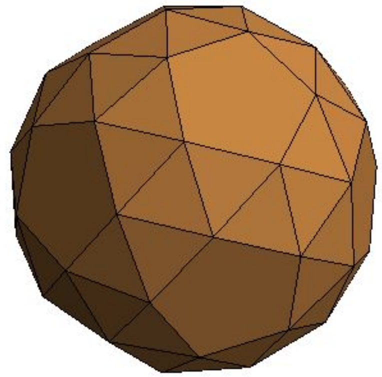
Truncated dodecahedron $d = 3$, $v = 60$,
$e = 90$, $\chi = 3$, $n \equiv 1$, 100, 136, 145 (mod 180)

Truncated icosidodecahedron $d = 3$, $v = 120$,
$e = 180$, $\chi = 2$, $n \equiv 1$, 136, 145, 280 (mod 360)
(Great rhombicosidodecahedron)

Snub cube $d = 5$, $v = 24$,
$e = 60$, $\chi = 3$, $n \equiv 1$, 16, 81, 96 (mod 120), $n \neq 16$

Residue class 1 (mod $2e$) done; TF & Griggs (2011+).

Archimedean graphs: other results

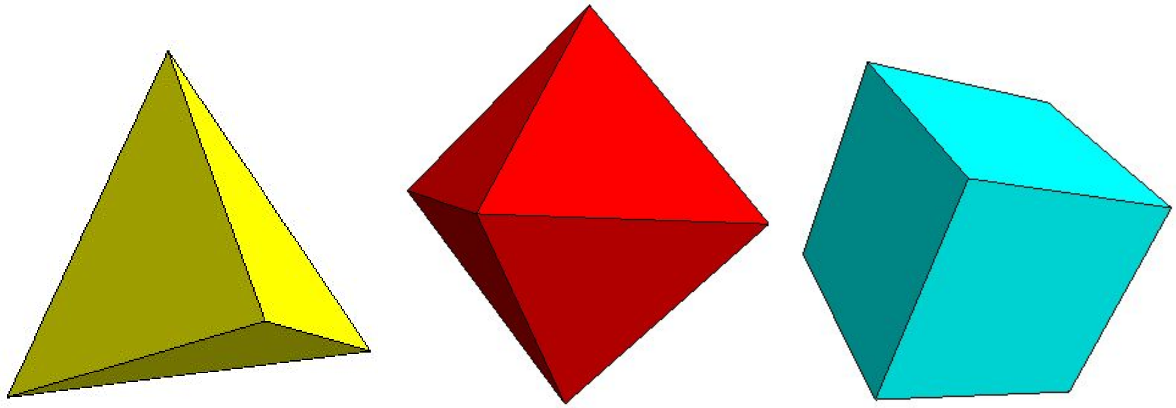
  

Icosidodecahedron $d = 4$, $v = 30$,
$e = 60$, $\chi = 3$, $n \equiv 1, 25, 81, 105 \pmod{120}$, $n \neq 25$
There exists a design of order 81; hence 81 (mod 240).

Truncated dodecahedron $d = 3$, $v = 60$,
$e = 90$, $\chi = 3$, $n \equiv 1, 100, 136, 145 \pmod{180}$
There exists a design of order 145.

Snub dodecahedron $d = 5$, $v = 60$,
$e = 150$, $\chi = 4$, $n \equiv 1, 76, 201, 276 \pmod{300}$
Nothing!

Platonic graphs

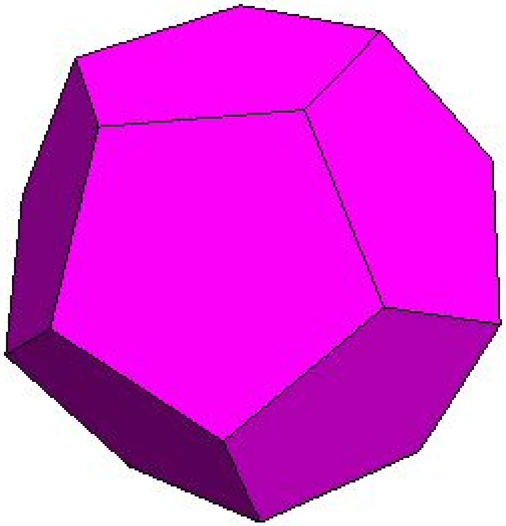


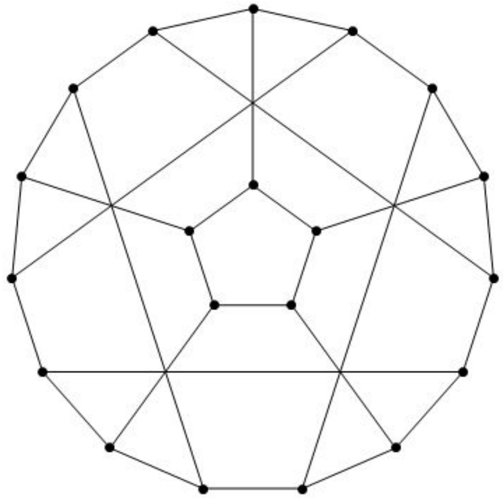
Tetrahedron $d = 3$, $v = 4$, $e = 6$, $\chi = 4$,
$n \equiv 1, 4 \pmod{12}$
Hanani (1961)

Octahedron $d = 4$, $v = 6$, $e = 12$, $\chi = 3$,
$n \equiv 1, 9 \pmod{24}$, $n \neq 9$
Griggs, deResmini & Rosa (1992);
Adams, Billington & Rodger (1994)

Cube $d = 3$, $v = 8$, $e = 12$, $\chi = 2$,
$n \equiv 1, 16 \pmod{24}$
Maheo (1980); Kotzig (1981);
Bryant, El-Zanati & Gardner (1994)

Platonic graphs





Dodecahedron $d = 3$, $v = 20$, $e = 30$, $\chi = 3$, $i = 8$,
$n \equiv 1, 16, 25, 40 \pmod{60}$, $n \neq 16$

Residue class 1 (mod 60): Adams & Bryant (1996)

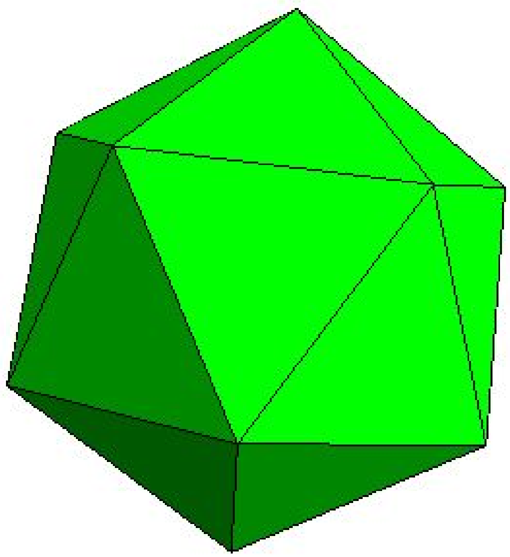
25, 40, 76: Adams, Bryant & Buchanan (2008)

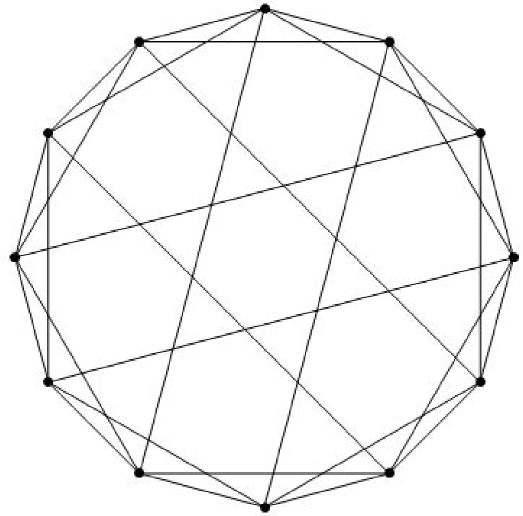
Completed: Adams, Bryant, TF & Griggs (2011)

Flower snark J5 $d = 3$, $v = 20$, $e = 30$, $\chi = 3$, $i = 9$,
$n \equiv 1, 16, 25, 40 \pmod{60}$, $n \neq 16$

Completed: TF

Platonic graphs





Icosahedron $d = 5$, $v = 12$, $e = 30$, $\chi = 4$, $i = 3$,
$n \equiv 1$, 16, 21, 36 (mod 60)

Residue class 1 (mod 60): Adams & Bryant (1996)

16, 76: Adams, Bryant & Buchanan (2008)

Completed except for 21, 141, 156, 201, 261, 276:
TF & Griggs (2011)

Transitive 5-regular 12-vertex graph $d = 5$, $v = 12$, $e = 30$, $\chi = 3$, $i = 4$,
$n \equiv 1$, 16, 21, 36 (mod 60)

Completed: TF

Icosahedron

THEOREM $\exists$ an icosahedron design of order $12jt + 4j + 15w + 1$ if $j \in \{15, 20\}$, $w \in \{0, 1, 4, 5, 8, 9, 12, 13\}$ and $t \geq w/4$.

j	w	min t	$12jt + 4j + 15w + 1$	exceptions
15	0	0	61, 241, 421, ...	–
15	4	1	301, 481, 661, ...	121
15	8	2	541, 721, 901, ...	1, 181, **361**
15	1	1	256, 436, 616, ...	76
15	5	2	496, 676, 856, ...	136, **316**
15	9	3	736, 916, 1096, ...	16, 196, **376**, **556**
20	0	0	81, 321, 561, ...	–
20	4	1	381, 621, 861, ...	**[141]**
20	8	2	681, 921, 1161, ...	**[201]**, **441**
20	12	3	981, 1221, 1461, ...	**[21]**, **[261]**, **501**, **741**
20	1	1	336, 576, 816, ...	**96**
20	5	2	636, 876, 1116, ...	**[156]**, **396**
20	9	3	936, 1176, 1416, ...	**216**, **456**, **696**
20	13	4	1236, 1476, 1716, ...	**36**, **[276]**, **516**, **756**, **996**

Group divisible designs

K-GDD of type u^t: triple (V, G, B), where
V is a set of size tu,
G is a partition of V into t subsets of size u (*groups*),
B is a set of subsets of sizes $k \in K$ (*blocks*) such that a pair of elements from different groups occurs in precisely one block but no pair of elements from the same group occurs at all.

K-GDD of type $u^t w^1$: t groups of size u and one group of size w.

k-GDD: $\{k\}$-GDD.

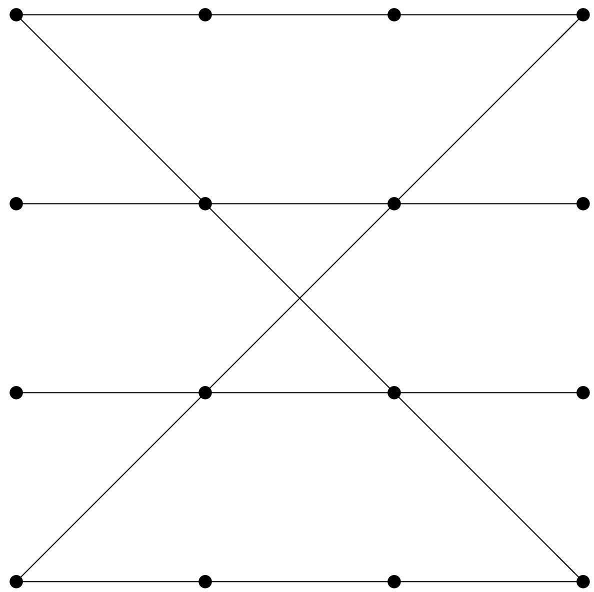
Parallel class: a subset of the block set in which each element of the base set appears exactly once.

Resolvable K-GDD (K-RGDD): the entire set of blocks can be partitioned into parallel classes.

Steiner system $S(2, k, v)$: (V, B) of a k-GDD of type 1^v.

Affine plane of order k: Steiner system $S(2, k, k^2)$ — resolvable.

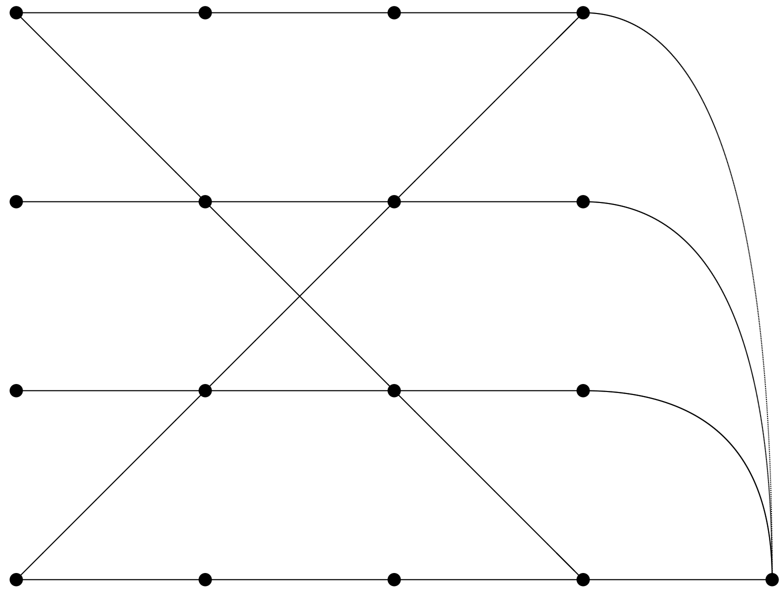
Proof of the theorem (i) Start with a 4-RGDD of type 4^{3t+1}



$\exists$ 4-RGDD of type 4^{3t+1}, Hanani, Ray-Chaudhuri & Wilson (1972).
$4t(3t+1)$ blocks resolvable into $4t$ parallel classes.

Picture represents the case $t = 1$.
Columns represent the groups (of size 4).
The horizontal lines form a parallel class.
Two other typical blocks shown.

Proof of the theorem (ii) Add a new group of w points.

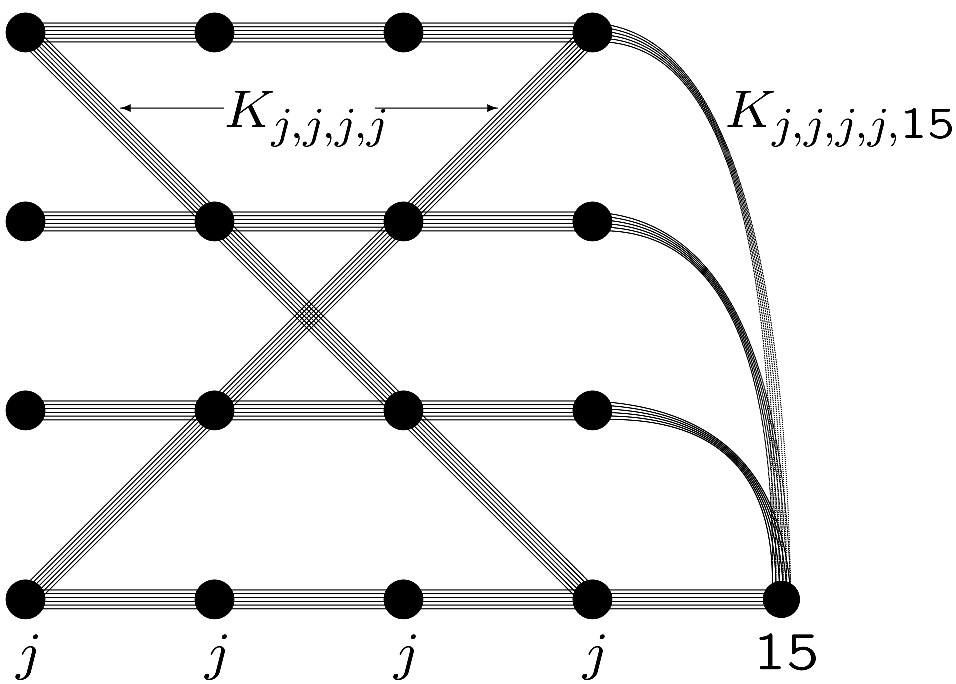


Add a new group of size w, $0 \leq w \leq 4t$, $w = 0$, 1, 4, 5, 8, 9, 12 or 13.
Map each new point to a distinct parallel class and extend its blocks to its associated point.

$w = 0$	$\Rightarrow$	4-GDD of type 4^{3t+1} (nothing new).
$0 < w < 4t$	$\Rightarrow$	$\{4,5\}$-GDD of type $4^{3t+1}w^1$.
$w = 4t$	$\Rightarrow$	5-GDD of type $4^{3t+1}w^1$.

$w = 1$ in the picture.
So four 5-element blocks are created.

Proof of the theorem (iii) Inflate.



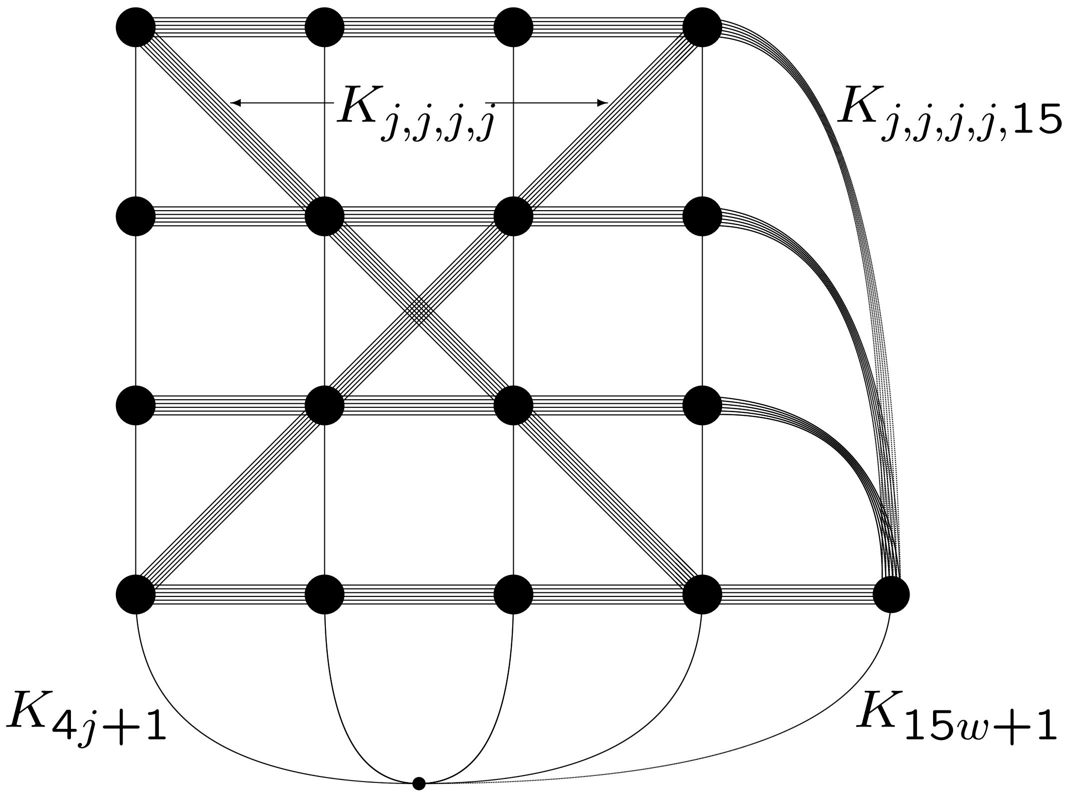
$w = 0$, 1, 4, 5, 8, 9, 12 or 13

Inflate points in the original 4-RGDD by j, $j = 15$ or 20.
Inflate points in the new group by 15.
Replace original 4-element blocks by $K_{j,j,j,j}$.
Replace new 5-element blocks by $K_{j,j,j,j,15}$.

Known icosahedron decompositions:
$K_{20,20,20,20}$, $K_{15,15,15,15,15}$ from Adams & Bryant (1996).

New icosahedron decompositions:
$K_{15,15,15,15}$, $K_{20,20,20,20,15}$.

Proof of the theorem (iv) Add a point and add the vertical graphs.



$j = 15$ or 20

$w = 0$, 1, 4, 5, 8, 9, 12 or 13

Overlay each original inflated group plus new point by K_{4j+1}.
Overlay each new inflated group plus new point by K_{15w+1}.

Known icosahedron decompositions:
K_{61}, K_{121}, K_{181} from Adams & Bryant (1996);
K_{16}, K_{76} from Adams, Bryant & Buchanan (2008).

New icosahedron decompositions: K_{81}, K_{136}, K_{196}.

Total points: $4j(3t + 1) + 15w + 1 = 12jt + 4j + 15w + 1$.

Icosahedron decompositions

Complete graphs:

K_{16}, K_{36}, K_{61}, K_{81}, K_{121}, K_{181}

Complete uniform multipartite graphs:

$K_{15,15,15,15}$, $K_{20,20,20,20}$, $K_{30,30,30,30}$, $K_{40,40,40,40}$,
$K_{15,15,15,15,15}$, $K_{30,30,30,30,30}$,
$K_{2,2,2,2,2,2}$, $K_{4,4,4,4,4,4}$, $K_{6,6,6,6,6,6}$, $K_{10,10,10,10,10,10}$,
$K_{12,12,12,12,12,12}$, $K_{20,20,20,20,20,20}$, $K_{36,36,36,36,36,36}$,
K_{10^7}, K_{20^7}, K_{15^8}, K_{15^9}, K_{20^9}, $K_{20^{10}}$, $K_{6^{11}}$, $K_{12^{11}}$,
$K_{18^{11}}$, $K_{5^{12}}$, $K_{10^{12}}$, $K_{15^{12}}$, $K_{5^{13}}$, $K_{10^{13}}$, $K_{15^{13}}$,
$K_{2^{16}}$, $K_{4^{16}}$, $K_{8^{16}}$, $K_{10^{18}}$, $K_{2^{21}}$, $K_{3^{21}}$, $K_{6^{21}}$,
$K_{5^{24}}$, $K_{5^{25}}$, $K_{2^{31}}$, $K_{4^{31}}$, $K_{3^{41}}$, $K_{2^{61}}$, $K_{2^{91}}$.

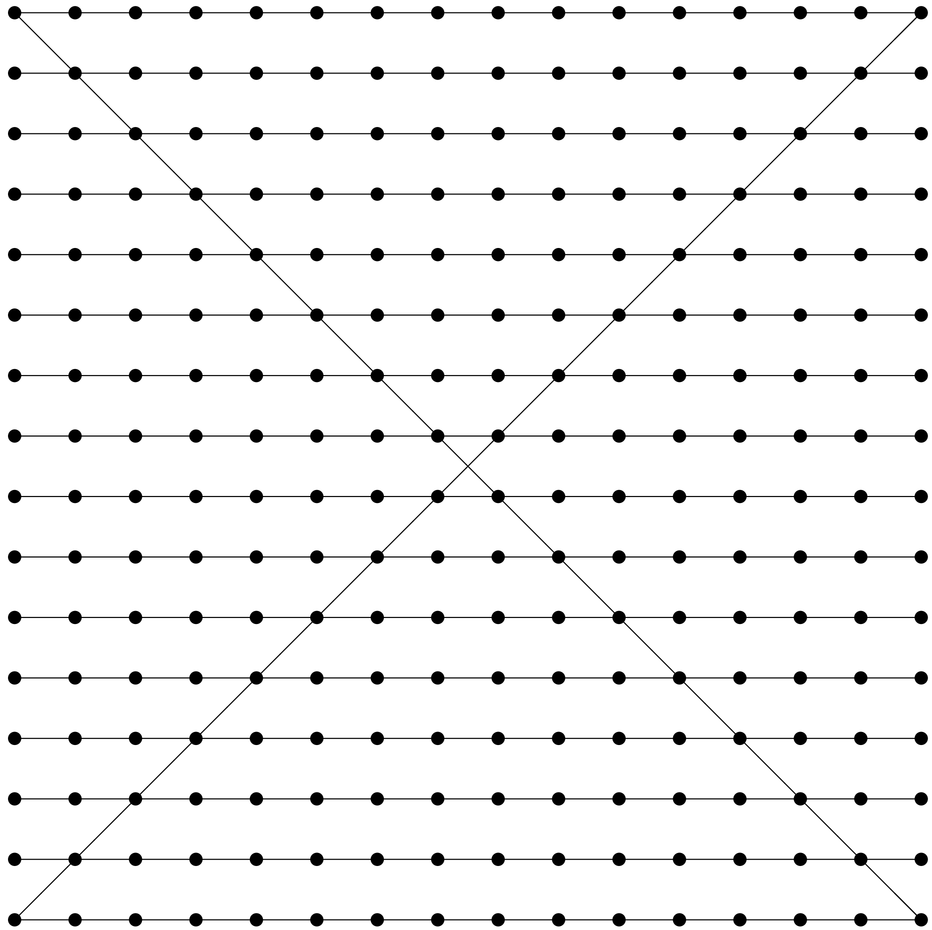
Complete non-uniform multipartite graphs:

$K_{20,20,20,20,15}$, $K_{20,20,20,20,30}$, $K_{30,30,30,30,15}$, $K_{40,40,40,40,30}$,
$K_{10^5,5}$, $K_{15^5,30}$, $K_{25^5,10}$, $K_{10^6,5}$, $K_{10^6,25}$, $K_{20^6,10}$,
$K_{10^7,30}$, $K_{5^9,10}$, $K_{10^9,20}$,
$K_{10^{10},15}$, $K_{5^{15},25}$, $K_{5^{17},20}$,
$K_{5^{19},15}$, $K_{5^{21},10}$, $K_{5^{31},15}$.

The one that got away:

K_{21}.

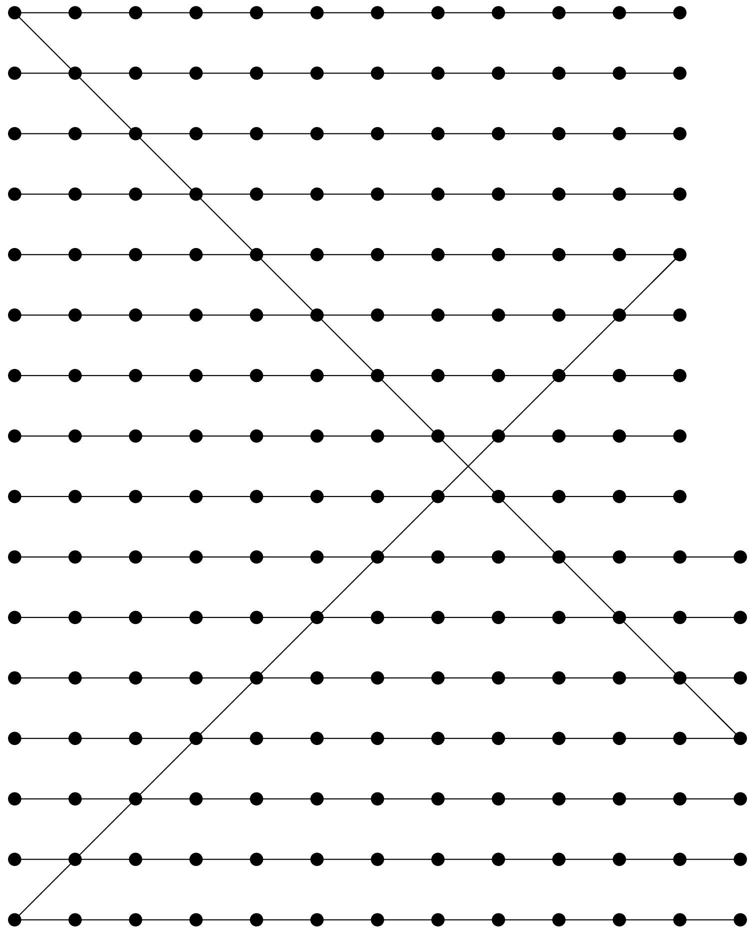
Icosahedron design of order 996 (i) Affine plane of order 16.



16 is a prime power.
Hence ∃ an affine plane of order 16.
Removing a parallel class creates a 16-GDD of type 16^{16}.

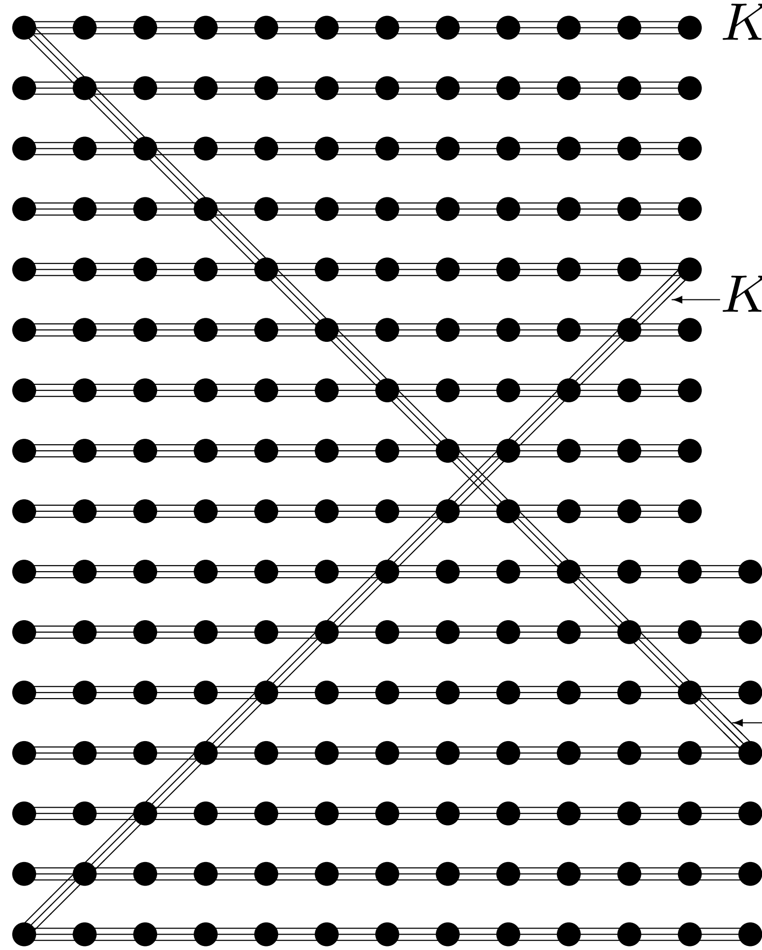
Picture shows the horizontal parallel class
plus two typical blocks.

Icosahedron design of order 996 (ii) Remove 3 groups & 9 points.



Remove 3 groups and 9 points from a single group
to give a $\{12, 13\}$-GDD of type $16^{12}7^1$.

Icosahedron design of order 996 (iii) Inflate by 5.



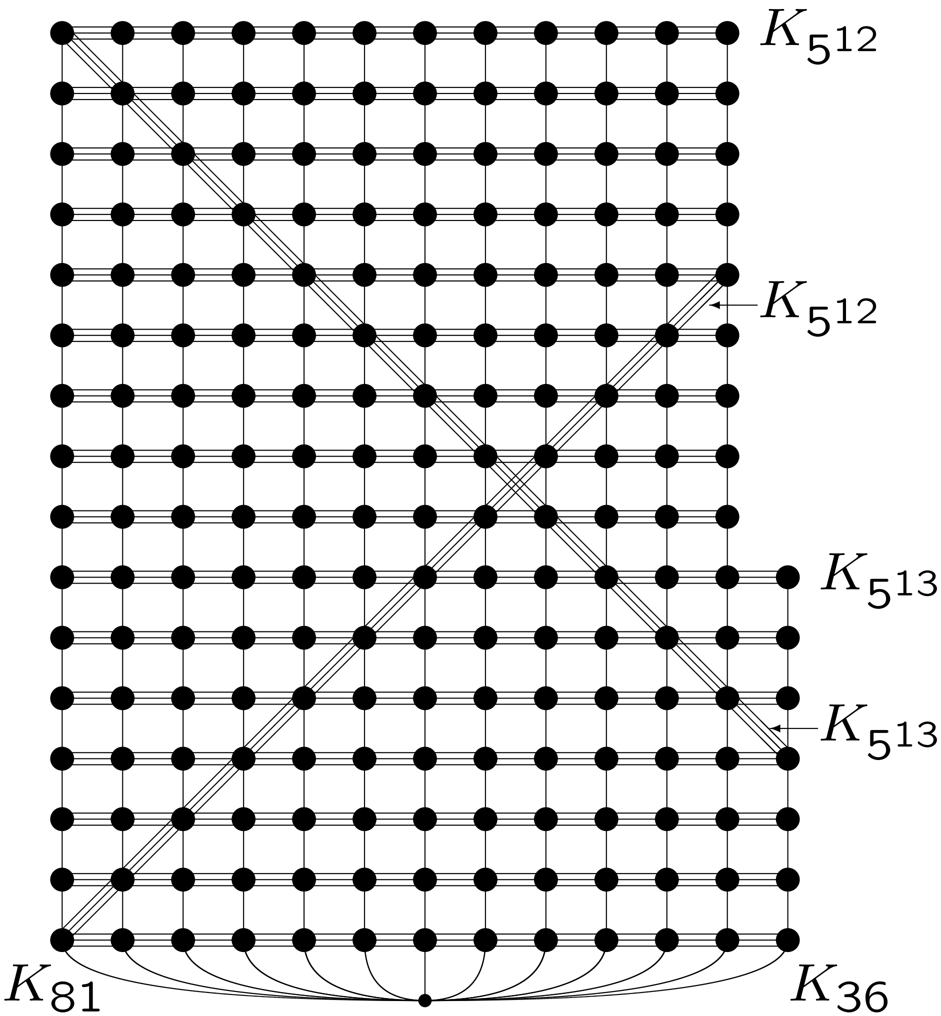
Inflate each point by a factor of 5.

Replace 12-element blocks by $K_{5^{12}}$.

Replace 13-element blocks by $K_{5^{13}}$.

New icosahedron decompositions: $K_{5^{12}}$, $K_{5^{13}}$.

Icosahedron design of order 996 (iv) Add a point; add vertical graphs.



Add a new point.

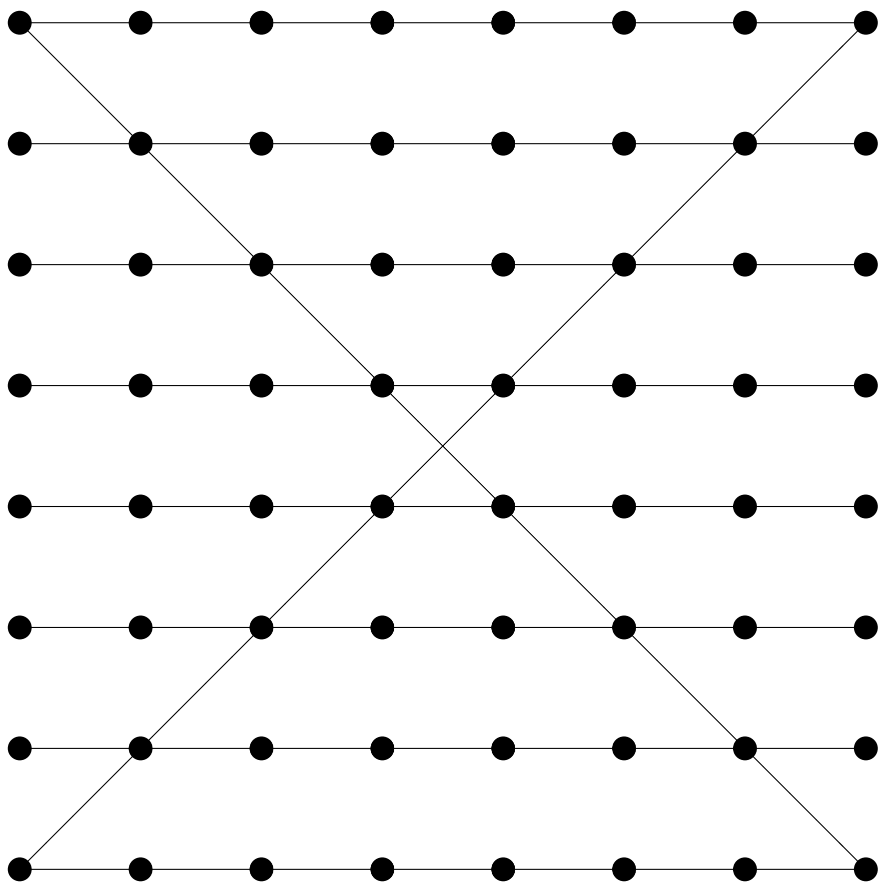
Overlay inflated 16-element group + new point by K_{81}.

Overlay inflated 7-element group + new point by K_{36}.

New icosahedron decompositions: K_{36}, K_{81}.

Total points: $12 \cdot 80 + 35 + 1 = 996$.

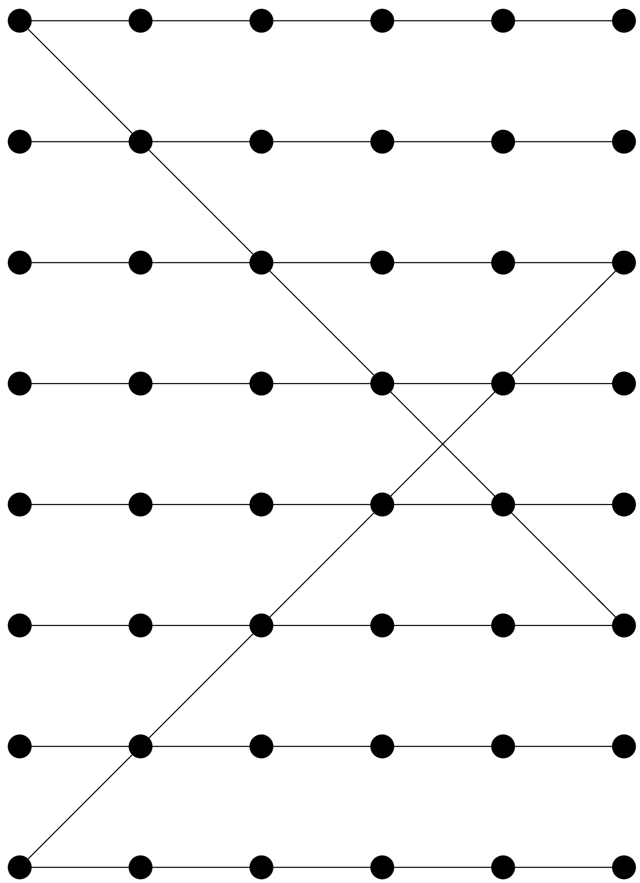
Icosahedron design of order 96 (i) Affine plane of order 8.



8 is a prime power.
Hence ∃ an affine plane of order 8.
Removing a parallel class ⇒ 8-GDD of type 8^8.

Picture shows the horizontal parallel class
plus two typical blocks.

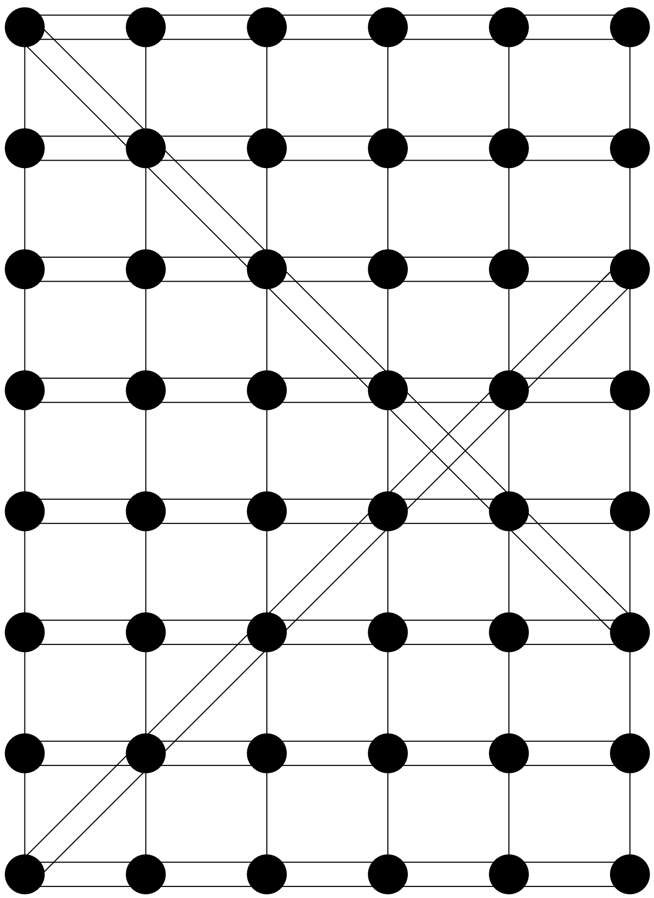
Icosahedron design of order 96 (ii) Remove 2 groups.



Remove 2 groups entirely to give a 6-GDD of type 8^6.

(Alternatively, construct this thing from four MOLS of side 8.)

Icosahedron design of order 96 (iii) Inflate by 2 and add graphs.



Inflate each point by a factor of 2.

Replace each block by $K_{2,2,2,2,2,2}$.

Replace each group by K_{16}.

Use icosahedron decompositions of K_{16}, $K_{2,2,2,2,2,2}$.

$6 \cdot 2 \cdot 8 = 96.$