Configurations and colouring problems in block designs

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Abstract

A Steiner triple system of order v (STS(v)) is called χ-*chromatic* if χ is the smallest number of colours needed to avoid monochromatic blocks. Amongst our results on colour class structures we show that every STS(19) is 3- or 4-chromatic, that every 3-chromatic STS(19) has an *equitable* 3-colouring (meaning that the colours are as uniformly distributed as possible), and that for all admissible $v \geq 25$ there exists a 3-chromatic STS(v) which does not admit an equitable 3-colouring. We obtain a formula for the number of independent sets in an STS(v) and use it to show that an STS(21) must contain eight independent points. This leads to a simple proof that every STS(21) is 3- or 4-chromatic.

Substantially extending existing tabulations, we provide an enumeration of STS trades of up to 12 blocks, and as an application we show that any pair of STS(15)s must be 3^{-1}-isomorphic.

We prove a general theorem that enables us to obtain formulae for the frequencies of occurrence of configurations in triple systems. Some of these are used in our proof that for $v \geq 25$ no STS(v) has a 3-existentially closed block intersection graph. Of specific interest in connection with a conjecture of Erdős are 6-sparse and perfect Steiner triple systems, characterized by the avoidance of specific configurations. We describe two direct constructions that produce 6-sparse STS(v)s and we give a recursive construction that preserves 6-sparseness. Also we settle an old question concerning the occurrence of perfect block transitive Steiner triple systems.

Finally, we consider Steiner $S(2, 4, v)$ designs that are built from collections of Steiner triple systems. We solve a longstanding problem by constructing such systems with $v = 61$ (Zoe's design) and $v = 100$ (the design of the century).